PMM U.S.S.R., Vol.53, No.2, pp.145-149, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

NON-LINEAR DYNAMICS OF A CONSERVATIVE SYSTEM DEGENERATING INTO A SYSTEM WITH A SINGULAR SET*

R.F.NAGAYEV and V.N. PILIPCHUK

A non-linear dynamical system with an arbitrary (but finite) number of degrees of freedom, which has a singular set in the degenerate case - a manifold of codimension one, is investigated. The characteristic feature of such systems, which describe the dynamics of various elastic flexible constructions, is that the degenerate system remains essentially non-linear of the same dimensions as the original system. This impedes direct construction of an asymptotic solution with respect to a suitable small parameter. As will be shown, the construction can be accomplished after some preliminary transformations, based on the idea of presenting the motion of the system as a drift of a localized region of high-frequency oscillations over the singular set, deviating from that set in the normal direction only.

The theory is illustrated by treating the problems involved in the oscillations of a thin-walled shallow arch and an elastic circular ring.

We consider mechanical systems described by a Lagrangian of the following form:

$$L = T - \Pi \tag{1}$$

$$T = {}^1/_2 x'^2, \quad \Pi = {}^1/_2 f^2(x) + \varepsilon^2 \Phi(x)$$
$$x = (x_1, \ldots, x_n) \tag{2}$$

where $\varepsilon^2 \ll 1$ is a parameter and $f(x)$ and $\Phi(x)$ are holomorphic functions, the first of which has the property that the set

$$M_f = \{x\colon f(x) = 0\} \tag{3}$$

is an $(n-1)$-dimensional manifold in R^n.

Such situations arise, for example, when one is investigating the dynamics of certain flexible elastic structures that may undergo large displacements in reaction to small relative deformations. In such cases the manifold M_f corresponds to the continuous set of equilibrium modes of the degenerate (absolutely flexible) structure /1, 2/.

Direct construction of solutions of the equations of motion corresponding to (1) and (2) is difficult, and in fact the smallness of the parameter ε is at first sight useless, since the degenerate system ($\varepsilon = 0$) remains essentially non-linear and of the same dimensions as the non-degenerate system. Moreover, analysis of the expression Π in (2) shows that, if the energy is given, the effect of the non-linearity increases as ε decreases. This is in agreement with common conceptions of the dynamics of flexible structures: if the energy of the oscillations is fixed, their amplitudes are higher the more flexible the structure.

We will reduce the system to a form that is easier to handle by standard methods; this will be done via a certain coordinate transformation.

Let $y = (y_1, \ldots, y_n)$ be an arbitrary point of M_f: $f(y) \equiv 0$. We shall assume that at least a sufficiently large part of the manifold admits of a parametrization in terms of a curvilinear orthogonal coordinate system $s = (s_1, \ldots, s_{n-1})$:

$$y = y(s), \quad \frac{\partial y}{\partial s_\alpha}\frac{\partial y}{\partial s_\beta} \equiv \frac{\partial y_i}{\partial s_\alpha}\frac{\partial y_i}{\partial s_\beta} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, n-1$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and repeated indices indicate summation.

*Prikl. Matem. Mekhan., 53, 2, 190-195, 1989

Let $n(s) = (n_1, \ldots, n_n)$ denote the unit vector normal to M_f at the point $y(s)$:

$$n_i = \frac{1}{\omega}\frac{\partial f}{\partial y_i}, \quad \omega^2 = \frac{\partial f}{\partial y_i}\frac{\partial f}{\partial y_i}, \quad i = 1, \ldots, n$$

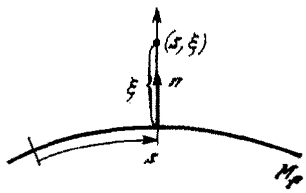

Fig.1

Let us assume that the coordinate ξ normal to M_f is such that the relationship of the coordinates (s, ξ) to the original coordinates (Cartesian in R^n) is as follows (Fig.1):

$$x = y + n\xi, \; y = y(s), \; n = n(s) \tag{4}$$

Then the expression for the kinetic energy (2) in the new coordinates is

$$T = {}^1/_2(\delta_{\alpha\beta} + 2m_{\alpha\beta}\xi + n_{\alpha\beta}\xi^2)s_\alpha^{\cdot}s_\beta^{\cdot} + {}^1/_2\xi^{\cdot 2}$$
$$m_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial y_i}{\partial s_\alpha}\frac{\partial n_i}{\partial s_\beta} + \frac{\partial y_i}{\partial s_\beta}\frac{\partial n_i}{\partial s_\alpha}\right), \quad n_{\alpha\beta} = \frac{\partial n_i}{\partial s_\alpha}\frac{\partial n_i}{\partial s_\beta}$$

For velocities $s_\alpha^{\cdot}$ we have generalized momenta

$$p_\alpha = \partial T/\partial s_\alpha^{\cdot} = s_\alpha^{\cdot} + (2m_{\alpha\beta}\xi + n_{\alpha\beta}\xi^2)s_\beta^{\cdot}$$

Conversely, $s_\alpha^{\cdot}$ can be expressed in terms of the momenta by expanding in powers of ξ (throughout, ξ is assumed to be sufficiently small)

$$s_\alpha^{\cdot} = p_\alpha - 2\xi m_{\alpha\beta}p_\beta - \xi^2 n_{\alpha\beta}' p_\beta + \ldots$$
$$n_{\alpha\beta}' = n_{\alpha\beta} - 4m_{\alpha\gamma}m_{\gamma\beta}, \quad \gamma = 1, \ldots, n-1$$

To transform to a standard system with one rapidly rotating phase, we use the Routh function $R = p_\alpha s_\alpha^{\cdot} - T + \Pi$.

Suppose the energy of the system is of the order of ε^2. Then it follows from (2) that the trajectories of the motion will lie in a certain neighbourhood of M_f with normal deviations ξ of order ε.

Expanding Π (see (2)) in powers of ξ and applying the following scale transformation of variables

$$\xi = \varepsilon\zeta, \; p_\alpha = \varepsilon r_\alpha \; (R = \varepsilon\bar{R})$$

we obtain

$$\bar{R} = \varepsilon(-{}^1/_2\zeta^{\cdot 2} + {}^1/_2\omega^2\zeta^2 + {}^1/_2 r_\alpha^2 + U_0 + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \ldots) \tag{5}$$
$$U^{(1)} = (U_1 + V_1\zeta^2 - m_{\alpha\beta}r_\alpha r_\beta)\zeta, \quad U^{(2)} = (U_2 + V_2\zeta^2 - {}^1/_2 n_{\alpha\beta}' r_\alpha r_\beta)\zeta^2$$
$$V_1(s) = \frac{\omega}{2}n_i n_j\frac{\partial^2 f}{\partial y_i\partial y_j}, \quad V_2(s) = \frac{1}{8}\left[n_i n_j\frac{\partial^2 f}{\partial y_i\partial y_j}\right]^2 + \frac{\omega}{6}n_i n_j n_k\frac{\partial^3 f}{\partial y_i\partial y_j\partial y_k}, \ldots$$
$$U_0(s) = \Phi(y), \quad U_1(s) = n_i\frac{\partial\Phi}{\partial y_i}, \quad U_2(s) = \frac{1}{2}n_i n_j\frac{\partial\Phi}{\partial y_i\partial y_j}, \ldots$$

System (5) is Lagrangian with respect to the normal coordinate ζ and Hamiltonian with respect to the variable s_α, r_α. The corresponding equations of motion are

$$\zeta^{\cdot\cdot} + \omega^2\zeta + \varepsilon\frac{\partial U^{(1)}}{\partial\zeta} + \varepsilon^2\frac{\partial U^{(2)}}{\partial\zeta} + \ldots = 0$$
$$s_\alpha^{\cdot} = \varepsilon\left(r_\alpha + \varepsilon\frac{\partial U^{(1)}}{\partial r_\alpha} + \varepsilon^2\frac{\partial U^{(2)}}{\partial r_\alpha} + \ldots\right)$$
$$r_\alpha^{\cdot} = -\varepsilon\left(\omega\frac{\partial\omega}{\partial s_\alpha}\zeta^2 + \frac{\partial U_0}{\partial s_\alpha} + \varepsilon\frac{\partial U^{(1)}}{\partial s_\alpha} + \varepsilon^2\frac{\partial U^{(2)}}{\partial s_\alpha} + \ldots\right) \tag{6}$$

At the initial time the new variables stand in the following relation to the old ones:

$$t = 0, \; \zeta = \varepsilon^{-1}(x_i - y_i)n_i, \; \zeta^{\cdot} = \varepsilon^{-1}x_i^{\cdot}n_i$$
$$r_\alpha = \frac{1}{\varepsilon}\left(x_i^{\cdot}\frac{\partial y_i}{\partial s_\alpha} + \varepsilon\zeta\frac{\partial y_i}{\partial s_\beta}\frac{\partial n_i}{\partial s_\alpha}\frac{\partial y_j}{\partial s_\beta}x_j^{\cdot} + \ldots\right)$$
$$[x_i - y_i(s)]\frac{\partial y_i(s)}{\partial s_\alpha} = 0$$

where, in view of the assumed order of magnitude of the energy, the right-hand sides of the equations for $\zeta, \zeta^{\cdot}, r_\alpha$ are of the order of unity, and the last equation yields a system of equations for the initial values of the coordinates s_α, $\alpha = 1, \ldots, n-1$.

We now introduce action-angle variables $(I - \varphi)$:

$$\zeta = \sqrt{2I/\omega}\cos\varphi, \quad \zeta^{\cdot} = -\sqrt{2I\omega}\sin\varphi \tag{7}$$

In view of the relation

$$\omega^{\cdot} = \frac{\partial\omega}{\partial s_\alpha} s_\alpha^{\cdot} = \varepsilon\frac{\partial\omega}{\partial s_\alpha}\left(r_\alpha + \varepsilon\frac{\partial U^{(1)}}{\partial r_\alpha} + \ldots\right)$$

we finally obtain a system with one fast phase:

$$\begin{gathered}
\varphi^{\cdot} = \omega + \varepsilon\left(\frac{\partial U^{(1)}}{\partial I} - \frac{r_\alpha}{2\omega}\frac{\partial\omega}{\partial s_\alpha}\sin 2\varphi\right) + \\
\varepsilon^2\left(\frac{\partial U^{(2)}}{\partial I} - \frac{1}{2\omega}\frac{\partial\omega}{\partial s_\alpha}\frac{\partial U^{(1)}}{\partial r_\alpha}\sin 2\varphi\right) + \ldots \\
I^{\cdot} = -\varepsilon\left(\frac{\partial U^{(1)}}{\partial\varphi} - \frac{I}{\omega}\frac{\partial\omega}{\partial s_\alpha} r_\alpha\cos 2\varphi\right) - \varepsilon^2\left(\frac{\partial U^{(2)}}{\partial\varphi} - \right. \\
\left.\frac{I}{\omega}\frac{\partial\omega}{\partial s_\alpha}\frac{\partial U^{(1)}}{\partial r_\alpha}\cos 2\varphi\right) - \ldots, \quad s_\alpha^{\cdot} = \varepsilon r_\alpha + \varepsilon^2\frac{\partial U^{(1)}}{\partial r_\alpha} + \ldots \\
r_\alpha^{\cdot} = -\varepsilon\left(\frac{\partial\omega}{\partial s_\alpha} I + \frac{\partial U_0}{\partial s_\alpha} + \frac{\partial\omega}{\partial s_\alpha} I\cos 2\varphi\right) - \varepsilon^2\frac{\partial' U^{(1)}}{\partial s_\alpha} - \ldots \\
U^{(1)} = A_1(I; s, r)\cos\varphi + B_1(I; s)\cos 3\varphi \\
U^{(2)} = A_2(I; s, r) + B_2(I; s, r)\cos 2\varphi + C_2(I; s)\cos 4\varphi \\
A_1 = \sqrt{\frac{2I}{\omega}}\left(U_1 + \frac{3I}{2\omega}V_1 - m_{\alpha\beta}r_\alpha r_\beta\right), \quad B_1 = \sqrt{\frac{I^3}{2\omega^3}}V_1 \\
A_2 = \frac{I}{\omega}\left(U_2 + \frac{3I}{2\omega}V_2 - \frac{1}{2}n'_{\alpha\beta}r_\alpha r_\beta\right) \\
B_2 = \frac{I}{\omega}\left(U_2 + \frac{2I}{\omega}V_2 - \frac{1}{2}n'_{\alpha\beta}r_\alpha r_\beta\right), \quad C_2 = \frac{I^2V_2}{2\omega^2}
\end{gathered} \tag{8}$$

(evaluation of the derivatives $\partial'/\partial s_\alpha$ does not require differentiation of the variable ζ).

After applying Bogolyubov's averaging procedure /3, 4/ to Eqs.(8), we obtain a system in which the fast phase does not occur on the right (two approximations):

$$\begin{gathered}
\bar\varphi^{\cdot} = \omega + \varepsilon^2\left\{\frac{\partial A_2}{\partial I} - \frac{1}{4\omega}\frac{\partial^2}{\partial I^2}(A_1^2 + B_1^2) - \right. \\
\left.\frac{1}{8\omega^3}\left[\left(\bar r_\alpha\frac{\partial\omega}{\partial\bar s_\alpha}\right)^2 - \bar I\omega\left(\left(\frac{\partial\omega}{\partial\bar s_\alpha}\right)^2\right)\right]\right\} + O(\varepsilon^3) \\
\bar I^{\cdot} = O(\varepsilon^3), \quad \bar s_\alpha^{\cdot} = \varepsilon\bar r_\alpha + O(\varepsilon^3), \quad \bar r_\alpha^{\cdot} = -\varepsilon\partial(\bar I\omega + U_0)/\partial\bar s_\alpha + O(\varepsilon^3)
\end{gathered} \tag{9}$$

The new variables $(\bar\varphi, \bar I; \bar s_\alpha, \bar r_\alpha)$ are related to the old ones by the formulae

$$\begin{gathered}
\varphi = \bar\varphi + \frac{\varepsilon}{\omega}\left(\frac{\partial A_1}{\partial I}\sin\bar\varphi + \frac{1}{3}\frac{\partial B_1}{\partial I}\sin 3\bar\varphi + \frac{\bar r_\alpha}{4\omega}\frac{\partial\omega}{\partial\bar s_\alpha}\cos 2\bar\varphi\right) + O(\varepsilon^2) \\
I = \bar I - \frac{\varepsilon}{\omega}\left(A_1\cos\bar\varphi + B_1\cos 3\bar\varphi - \frac{I}{2\omega}\frac{\partial\omega}{\partial\bar s_\alpha}\bar r_\alpha\sin 2\bar\varphi\right) + O(\varepsilon^2) \\
s_\alpha = \bar s_\alpha + O(\varepsilon^2), \quad r_\alpha = \bar r_\alpha - \varepsilon\frac{I}{2\omega}\frac{\partial\omega}{\partial\bar s_\alpha}\sin 2\bar\varphi + O(\varepsilon^2)
\end{gathered}$$

Thus, the original conservative system, perturbed by a quantity of the order of ε^2, can be reduced to a "quasilinear" system and studied by asymptotic expansion in powers of ε.

It should be noted that since $\partial\omega/\partial s_\alpha \neq 0$, the transformation (7) is not canonical. The structure of Eqs.(8) is therefore faulty. The essential point is that in the type of problem under consideration it seems impossible to derive exact Hamiltonian equations with one fast phase and the other variables slowly varying. It is all the more significant that up to quantities of the order of ε^2 the new action variable $\bar I$ in the averaged system (9) is an adiabatic invariant, while the equations for $\bar s_\alpha$ and $\bar r_\alpha$ have a canonical structure and therefore have a first integral

$$\tfrac{1}{2}\bar r_\alpha^2 + \bar I\omega + U_0 = h \quad (h = \mathrm{const}) \tag{10}$$

If the system has two degrees of freedom, the manifold M_f of (3) is a curve in the coordinate plane x_1x_2, so that there remains only one coordinate $\bar s_1 \equiv S$; using (10) one can express the latter by quadrature as a function of time:

$$\int_{S^\circ}^{S} \frac{dS}{\sqrt{2[h - I\omega(S) - U_0(S)]}} = \varepsilon t; \quad S^\circ = S|_{t=0} \tag{11}$$

Thus, in the case of two degrees of freedom the problem of integrating the equations of motion reduces to evaluation of an integral.

Example 1. The equation of the free non-linear oscillations of a thin-walled shallow arch when ends clamped /5/ may be written, after an appropriate scale transformation of the variables, as

$$\frac{\partial^2 W}{\partial \tau^2} + \varepsilon^2 \frac{\partial^4 (W - W_0)}{\partial \eta^4} - P \frac{\partial^2 W}{\partial \eta^2} = 0 \quad (0 \leqslant \eta \leqslant \pi)$$

$$P = \frac{1}{2\pi} \int_0^{\pi} \left[\left(\frac{\partial W}{\partial \eta} \right)^2 - \left(\frac{\partial W_0}{\partial \eta} \right)^2 \right] d\eta, \quad \varepsilon^2 = \frac{1}{e^2} \frac{EI}{EA_s}$$

where e is a parameter representing the lifting power of the arch, P is the thrust, $W(\eta, \tau)$ is the coordinate of the elastic curve, measured in fractions of e, τ is a time parameter.

For a two-hinged sinusoidal arch, putting

$$W_0 = -\sin\eta, \quad W = x_1(\tau)\sin\eta + x_2(\tau)\sin 2\eta$$

we obtain a coupled system of two non-linear equations in the coefficients of the first two arch modes:

$$d^2x_1/d\tau^2 + \varepsilon^2(x_1 + 1) + {}^1/_4(x_1^2 + 4x_2^2 - 1)x_1 = 0$$

$$d^2x_2/d\tau^2 + 16\varepsilon^2 x_2 + (x_1^2 + 4x_2^2 - 1)x_2 = 0$$

where the coefficient of the linear terms is a small parameter, so that direct construction of an asymptotic solution is a difficult task.

In the plane of the frist two modes, expressions for the kinetic and potential energy of the elastic deformations are easily determined from the form of the equations of motion:

$$T = {}^1/_2(x_1^{\cdot 2} + x_2^{\cdot 2})(^{\cdot} \equiv d/d\tau), \quad \Pi = {}^1/_2 f^2 + \varepsilon^2\Phi \tag{12}$$

$$f = {}^1/_4\sqrt{2}(x_1^2 + 4x_2^2 - 1), \quad \Phi = {}^1/_2(x_1 + 1)^2 + 8x_2^2$$

The manifold $f(x_1, x_2) = 0$ is in this case an ellipse in the x_1x_2 coordinate plane - the locus of configurations of the arch with no extension-compression deformations of the central curve. **Displacement along this** ellipse corresponds to pure deflections of the arch.

Expressing the equation of the ellipse as

$$x_1 = y_1 = -\cos\theta, \quad x_2 = y_2 = {}^1/_2\sin\theta$$

we obtain $\omega = [{}^1/_2(1 + 3\sin^2\theta)]^{1/2}$, $dS^2 = {}^1/_2\omega^2 d\theta^2$. Eq.(11) becomes

$$\int_{\theta_0}^{\theta} \frac{\omega\, d\theta}{2\sqrt{h - F(\theta)}} = \varepsilon t, \quad \theta_0 = \theta|_{t=0}$$

$$F(\theta) = I\omega(\theta) + {}^1/_2(5 - 2\cos\theta - 3\cos^2\theta)$$

The quantities necessary for the computation may be written as

$$U_1 = \frac{\sqrt{2}}{2\omega}(16 - \cos\theta - 15\cos^2\theta), \quad U_2 = \frac{1}{4\omega^2}(64 - 63\cos^2\theta)$$

$$V_1 = \frac{\sqrt{2}}{8\omega}(16 - 15\cos^2\theta), \quad V_2 = \frac{1}{64\omega^4}(16 - 15\cos^2\theta)^2$$

Fig.2 shows curves of $F(\theta)|_{I=0}$ (curve *1*), $\omega(\theta)$ (curve *2*), $\cos\theta^* = -{}^1/_3$. **For** $I \neq 0$ **one has** $F(\theta) = F(\theta)|_{I=0} + I\omega(\theta)$, **and** hence it follows that the high-frequency oscillations associated with extension-compression deformations (the term $I\omega$ in the expression for $F(\theta)$) increase the potential barrier between the initial $(\theta = 0)$ and "reversed" $(\theta = \pi)$ equilibrium positions of the arch and therefore make the arch more stable to buckling. We emphasize that we are concerned here with thin-walled arches $(\varepsilon^2 \ll 1)$, in which buckling is accompanied by bulging of skew-symmetric form.

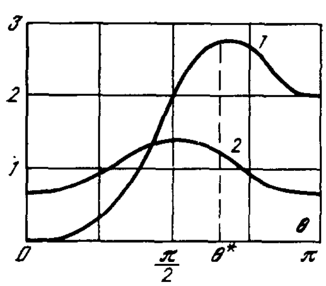

Fig.2

Example 2. Consider a thin-walled circular ring. Within the limits of flat-shell theory, the equation of the free oscillations of the ring may be written as

$$\frac{\partial^2 W}{\partial \tau^2} + \varepsilon^2 \frac{\partial^4 W}{\partial \eta^4} - P\left(\frac{\partial^2 W}{\partial \eta^2} - 1\right) = 0 \quad (0 \leqslant \eta \leqslant 2\pi)$$

$$P = \frac{1}{2\pi} \int_0^{2\pi} \left[W + \frac{1}{2}\left(\frac{\partial W}{\partial \eta}\right)^2\right] d\eta$$

where the parameter ε is proportional to the relative thickness of the ring; W is measured along the outer normal from the non-deformed position of the central curve.

Setting

$$W = x_1 + \sqrt{2} x_2 \cos 2\eta$$

and proceeding as in the previous example, we obtain (12) with

$$f = x_1 + 2x_2^2, \quad \Phi = 8x_2^2$$

Thus, the manifold M_f in this problem is a parabola in the x_1x_2 plane. The position of a point on the parabola is uniquely defined by the coordinate y_2, in terms of which we express the quantities needed for the computation:

$$\omega = (1 + 16y_2^2)^{1/2}, \quad dS^2 = \omega^2 dy_2^2$$

$$\int_{y_2^0}^{y_2} \frac{\omega(y_2)\, dy_2}{\sqrt{2[h - F(y_2)]}} = \varepsilon t, \quad F(y_2) = I\omega + 8y_2^2$$

$$U_1 = \frac{64}{\omega} y_2^2, \quad U_2 = \frac{128}{\omega^2} y_2^2, \quad V_1 = \frac{32}{\omega} y_2^2, \quad V_2 = \frac{512}{\omega^4} y_2^4$$

We remark that a ring, unlike an arch, cannot buckle, and the effect of high-frequency oscillations with extension and compression of the central curve is merely to increase the effective flexural stiffness of the ring.

REFERENCES

1. MANEVICH L.I. and PILIPCHUK V.N., Non-linear oscillations of a three-link mechanical structure with several equilibrium positions. Prikl. Mekh., 17, 2, 1981.
2. PILIPCHUK V.N., On a method of investigating non-linear problems in the dynamics of rectangular plates with initial irregularities. Prikl. Mekh., 22, 2, 78-85, 1986.
3. BOGOLYUBOV N.N. and MITROPOL'SKII YU.A., Asymptotic Methods in Non-linear Oscillation Theory, Fizmatgiz, Moscow, 1963.
4. MOISEYEV N.N., Asymptotic Methods of Non-linear Mechanics, Nauka, Moscow, 1981.
5. VOL'MIR A.S., Non-linear Dynamics of Plates and Shells, Nauka, Moscow, 1972.

Translated by D.L.